Exact discrete breather compactons in nonlinear Klein-Gordon lattices

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We demonstrate the existence of exact discrete compact breather solutions in nonlinear Klein-Gordon systems, and complete the work of Tchofo Dinda and Remoissenet [Phys. Rev. E **60**, 6218 (1999)], by showing that the breathers stability is related principally to the lattice boundary conditions, the coupling term, and the harmonicity parameter.

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I. INTRODUCTION

Many physical phenomena involve some localization of energy in space. The formation of vortices in hydrodynamics, self-focusing in optics or plasmas, the formation of dislocations in solids under stress, and self-trapping of energy in proteins are some examples of many branches of science [1] where stable structures called solitons emerge. This type of wave with exceptional properties however presents some wings or tails at infinity. Indeed, in nonlinear optical fibers, the long-distance interaction between two entities leads to a strict debit limitation [2]. But recently, the concept of compactification or strict localization of solitary waves appeared in literature [3,4]. Indeed, Rosenau and Hyman [4] showed that solitary-wave solutions may be compactified under the influence on nonlinear dispersion which is capable of causing deep qualitative changes in nature of nonlinear phenomena. Such robust solitonlike solutions, characterized by the absence of an infinite tail or wing and whose width is velocity independent, have been called compactons [4]. The interaction properties of two compact entities or compactons may be compared to that of two hard spheres, i.e., without long-distance interaction. Dusuel *et al.* [5] have demonstrated the existence of static compacton in a real physical system, made up of identical pendulums connected by springs. The existence of compactlike kinks in nonlinear Klein-Grodon lattices with ϕ^4 on-site substrate potential, requiring the presence of nonlinear dispersion and absence of linear dispersion has been shown by Tchofo Dinda *et al.* [6]. Moreover, exact compactlike kink and pulse solutions of such systems have been also demonstrated [7]. Thus, an understanding of physical mechanisms which give rise to compactons is essential for predicting the conditions in which real physical systems can support such compact structures. Recently, some studies predicted the existence of breather compactons [3,8]. Indeed, Kivshar [8] showed that breathers with compact support may exist in a lattice of identical particles interacting via a purely anharmonic coupling, without any on-site substrate potential. This work has been completed by Tchofo Dinda and Remoissenet [9], by adding a soft ϕ^4 on-site substrate potential. Tchofo Dinda and Remoissenet have also shown that an exact breather compacton solution exits in the continuous limit, and a certain degree of harmonicity in the substrate potential is required to stabilize the breather compacton solution. These very important results raise a fundamental question:

Do exact discrete compactlike breather solutions exist in such lattices, i.e., a very stable solution with a long-time coherent duration, and a weak spatial extent? The purpose of the present paper is to demonstrate by using a pseudoinverse procedure [7,10–12] that, exact discrete compactlike breather solutions exist, in a soft ϕ^4 substrate potential, and that the breather stability is related to the boundary conditions of the lattice.

The paper is organized as follow. First, we present the lattice model and show analytically that it can admit exact compactlike breather solutions if the ϕ^4 potential is adequately chosen. Then, in Sec. III, we study numerically the stability of such compactlike breathers with fixed and free boundary conditions. Finally, Sec. IV is devoted to concluding remarks.

II. MODEL AND EQUATION OF MOTION

We consider a lattice model where a system of atoms with unit mass, coupled anharmonically to their nearest neighbors and interact with a nonlinear substrate potential $V(u_n)$. The Hamiltonian of the system is given by

$$H=\sum_n \frac{1}{2}\dot{u}_n^2+\frac{1}{4}K_{nl}(u_n-u_{n-1})^4+V(u_n), \tag{1}$$

where u_n is the scalar dimensionless displacement of the nth particle, and K_{nl} is a parameter controlling the strength of the nonlinear coupling. $V(u_n)$ is an on-site substrate potential that we will determine by using a pseudoinverse procedure [7,10–12], to obtain breathers compacton of the desired shape. It is also important to note that no linear coupling term is present in Eq. (1). Indeed, the presence of such a term gives rise to a phonon band which may enter directly in resonance with the internal modes of a compacton, leading to energy radiation from the compacton.

The equation of motion of the nth atom of the lattice is then given by

$$\ddot{u}_n=K_{nl}[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3]+F(u_n), \tag{2}$$

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where $F(u_n) = -dV(u_n)/du_n$, is the substrate force deriving from the substrate potential $V(u_n)$.

Assuming that the compactlike or breather compacton solution of Eq. (2) writes under the form

$$u_n = A_0 \phi(n) \theta(t), \tag{3}$$

with a spatial cosine shape, that is:

$$\begin{aligned} \phi_n &= \cos(s), \quad \text{if} \quad s \in [-\pi/2, +\pi/2], \\ \phi_n &= 0, \quad \text{if} \quad s \in]-\infty, -\pi/2[, \\ \phi_n &= 0, \quad \text{if} \quad s \in]+\pi/2, +\infty[, \end{aligned} \tag{4}$$

with $s = \xi n$, and where ξ is a constant corresponding to a discrete parameter, and A_0 the solution amplitude. $\theta(t)$ is a function which defines the breather behavior in time.

Now, following a pseudoinverse procedure, we first insert Eq. (4) in Eq. (3) then Eq. (3) in Eq. (2), in order to calculate the expression of $F(u_n)$.

Thus,

$$\frac{d^2 u_n}{dt^2} = A_0 \phi_n \ddot{\theta}, \tag{5}$$

$$\begin{aligned} \Delta = (u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3 = A_0^3 \theta^3 [(\phi_{n+1} - \phi_n)^3 \\ - (\phi_n - \phi_{n-1})^3], \end{aligned} \tag{6}$$

$$\begin{aligned} (\phi_{n+1} - \phi_n)^3 &= [\cos(s+\xi) - \cos(s)]^3 \\ &= (\cos s \cos \xi - \sin s \sin \xi - \cos s)^3, \end{aligned} \tag{7}$$

$$\begin{aligned} (\phi_n - \phi_{n-1})^3 &= [\cos(s) - \cos(s-\xi)]^3 \\ &= (\cos s - \cos s \cos \xi - \sin s \sin \xi)^3. \end{aligned} \tag{8}$$

Setting $A = -\sin s \sin \xi$, and $B = \cos s(\tau - 1)$, with $\tau = \cos \xi$, the difference of the two cubic difference writes

$$\begin{aligned} \Delta = 2A_0^3 \theta^3 B(B^2 + 3A^2) = 2A_0^3 \theta^3 [4\phi_n^3 (\tau-1)^2 (\tau + 1/2) \\ + 3\phi_n (\tau - 1)(1 - \tau^2)]. \end{aligned} \tag{9}$$

Thus, the substrate force deriving from the substrate potential writes

$$\begin{aligned} F(u_n) = A_0 \phi_n \ddot{\theta} - K_{nl} [8u_n^3 (\tau-1)^2 (\tau + 1/2) \\ - 6u_n A_0^2 \theta^2 (\tau-1)^2 (\tau+1)]. \end{aligned} \tag{10}$$

One remarks that expression (10) contains yet explicitly the function of time θ, and its second derivative.

Seeking a ϕ^4 potential structure, it seems natural to express the differential relation in θ, as a function proportional to u_n, that is

$$A_0 \phi_n \ddot{\theta} + 6K_{nl} A_0^2 u_n (\tau - 1^2)(\tau+1)\theta^2 = -\lambda u_n. \tag{11}$$

Replacing u_n by its expression (3), and setting $\mu = 6K_{nl} A_0^2 (\tau-1)^2 (\tau+1)$, Eq. (11) becomes

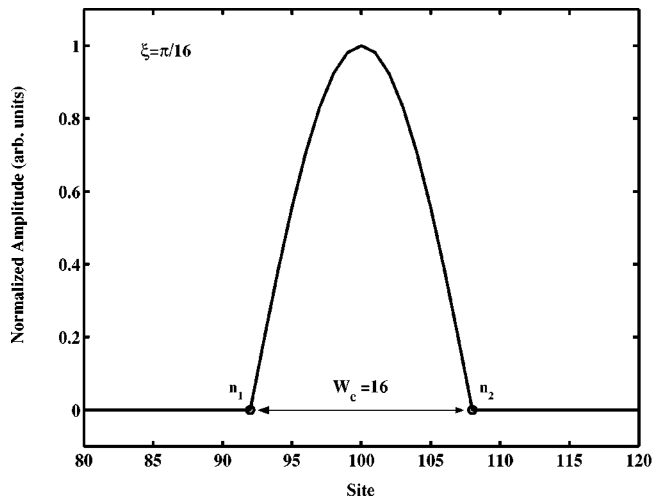

FIG. 1. Spatial sketch of the breather compacton solution at time $t=0$, for a discrete parameter $\xi = \pi/16$. The breather width is equal to $W_c = 16$ sites.

$$\ddot{\theta} = -(\lambda \theta + \mu \theta^3). \tag{12}$$

Equation (12) has the structure of a well-known equation which has for a solution

$$\theta = \mathrm{cn}(\omega t, k^2), \tag{13}$$

where cn is a Jacobian elliptic function of modulus $k^2 = \mu/2(\lambda + \mu)$, and pulsation $\omega = \sqrt{\lambda + \mu}$.

Thus, the lattice substrate force writes under the form

$$F(u_n) = -\frac{1}{2} \omega_0^2 (\alpha u_n + u_n^3), \tag{14}$$

with $\omega_0^2 = 16(\tau-1)^2(\tau + 1/2) K_{nl}$, and $\alpha = 2\lambda/\omega_0^2$.

The solution of Eq. (2) calculated for zero initial (at time $t=0$) velocity on the particles is then given by

$$\begin{aligned} u_n = A_0 \cos[\xi(n - n_0)] \mathrm{cn}(\omega t, k^2), \\ \text{if} \quad |n - n_0| < \frac{\pi}{2\xi} \quad \text{and} \quad u_n = 0, \end{aligned} \tag{15}$$

and presents a spatial width,

$$W_c = \pi/\xi. \tag{16}$$

The shape of such a solution is represented by Fig. 1 (at $t = 0$ for a discrete parameter $\xi = \pi/16$). Solution (15) may be compared to Eq. (5) of [9] obtained by Tchofo Dinda and Remoissenet in the continuous limit, and explains the remarkable stability of their simulations from the continuous to the discrete regime for important worths of the harmonicity parameter α, since both solutions have the same structure.

Note that, Eq. (2) is satisfied almost everywhere in the lattice except at the entity feet or nodes n_1 and n_2 like those represented in Fig. 1.

Indeed these two spatial singularities are at the origin of nonlinear oscillations emergence, and therefore to the de-

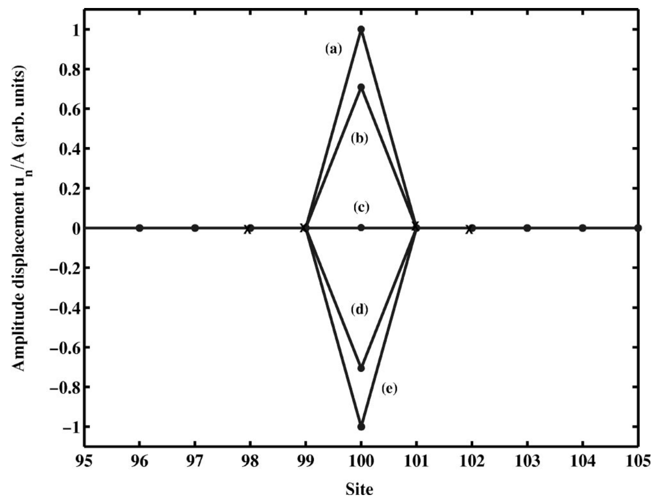

FIG. 2. Temporal breather dynamics over one period, in the ultimate degree of discretization: $\xi=\pi/2$. Other parameters $A_0=1$, $K_{nl}=0.01$, and $\alpha=10$. (a) corresponds at time $t=0$ and $t=T$, (b) time $t=T/8$ and $t=7T/8$, (c) time $t=T/4$ and $t=3T/4$, (d) time $t=3T/8$ and $t=5T/8$, and (e) time $t=T/2$ with [T (arb. units)]. Crosses (✕) represent the boundary conditions (zero displacement and zero velocity).

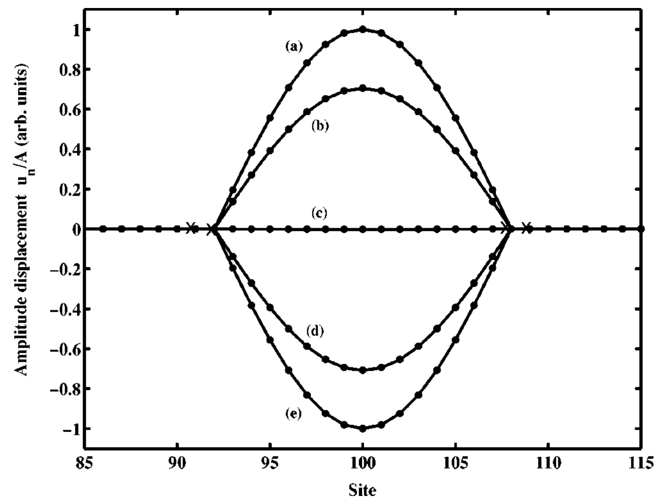

FIG. 3. Temporal breather dynamics over one period, in the continuous limit: $\xi=\pi/16$. Other parameters $A_0=1$, $K_{nl}=1$, and $\alpha=10$. (a) corresponds at time $t=0$ and $t=T$, (b) time $t=T/8$ and $t=7T/8$, (c) time $t=T/4$ and $t=3T/4$, (d) time $t=3T/8$ and $t=5T/8$, and (e) time $t=T/2$, with [T (arb. units)]. Crosses (✕) represent the boundary conditions (zero displacement and zero velocity).

struction of the breather structure, as it has been observed numerically by Tchofo Dinda and Remoissenet [9].

Moreover, it is clear that such a wave structure with spatial singularities is not physically correct, except if we fix boundary conditions with zero displacement and velocity, like for classical stationary waves on a rope fixed at its extremities.

III. NUMERICAL SIMULATIONS: BREATHER DYNAMICS

In the previous section, we have shown that Eq. (2) admits for solution expression (15). Here, we verify numerically the exact character of this solution for fixed boundary conditions, that is with zero displacement ($u_n=0$) and zero velocity ($\dot{u}_n=0$) at the breather feet or singular points, like those located by n_1 and n_2 in Fig. 1 or crosses in Figs. 2 and 3. The lattice size is equal to 200 sites.

Figures 2 and 3 represent, respectively, the temporal evolution of the compactlike breather profile over one period for the discrete parameters, and coupling terms, $\xi=\pi/2$, $K_{nl}=0.01$, and $\xi=\pi/16$, $K_{nl}=1$, with an harmonicity parameter $\alpha=10$. Note that, under these conditions, the breather profile is still conserved after a large number of oscillation periods (>100) and is accurate to the analytical predictions located by dots in Figs. 2 and 3. A systematic investigation of the breather behavior (not presented here) reveals that the stability of the entity is independent of the amplitude A_0, discrete parameter ξ, and harmonicity parameter α.

To obtain a stable breather with free boundary conditions, important values of the harmonicity parameter ($\alpha>100$) are necessary, like it has been already shown by Tchofo Dinda and Remoissenet [9]. Therefore, one can deduce that the breather stability is related to the boundary conditions. Indeed, our numerical simulations with free boundary conditions (with the same parameters as for fixed boundary conditions, and not represented here), reveal a total destruction of the initial breather structure before reaching one oscillation period at least, and show that the instabilities start to spread out at the feet of the breather (node n_1 and n_2 of sketch Fig. 1), that is at the singular points, by the emergence of standing phonons which interact with the core of the breather. Indeed, when we postulate or research a solution under the form (15), the boundary conditions are implicit, and suggest that the breather size be equal to the lattice size or vice versa.

These remarks are confirmed when important coupling worths leading to the continuum limit reduce strongly the singularities, like the small breather amplitudes. Note that the big values of the harmonicity parameter which favor the oscillatory term in comparison with the coupling term, tend also to stabilize the breather structure.

IV. CONCLUSION

In summary, we have demonstrated the existence of an exact discrete compact breather solution in a standard nonlinear Klein-Gordon system, and that the stability of this one is related to the boundary conditions of the lattice. Indeed, to complete the Tchofo Dinda and Remoissenet observations, and confirm the exact character of solution (15), we achieved any simulations with fixed boundary conditions that is with zero displacement ($u_n=0$) and zero velocity ($\dot{u}_n=0$) at the breather feet.

Others simulations not represented here allowed us to confirm that the solution singularities are at the origin of the breather instabilities. These instabilities may be toned down with a strong coupling (K_{nl}) that is in the continuous limit, or with small breather amplitude (A_0) for which the singularities are also reduced. Furthermore, important worths of the harmonicity parameter which favor the oscillatory term in comparison with the coupling term, tend also to stabilize the breather structure, as it has been shown numerically by Tchofo Dinda and Remoissenet. Note that, the breathers lifetime without instabilities (with free boundary conditions) is wide enough to present a genuine physical interest. Finally, it emerges from this paper that exact discrete compact breather solutions exist in nonlinear Klein-Gordon systems, and that the concept of compactification of solitary waves implies strict conditions that must be satisfied in order for physical systems to support localized modes with compact support. Some of those conditions have been explained qualitatively in the present paper. Nevertheless, in the actual stage of the research on structures with compact support, the results that have been obtained are still far away from practical applications, and much work remains to be done, in particular for the research of solutions without singularities effect, or particular potentials which cancel this effect. This effort deserves to be carried out to make the compacton concept a reality in some areas in which compactons could ensure practical applications such as in signal processing and communications.

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